BOUNDS FOR COLLATZ CYCLES

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Abstract. We give upper bounds for positive Collatz cycles in terms of their lengths. As a corollary, we see that the number of positive Collatz cycles of a given length is finite.

The *Collatz function* $C : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$C(a) := \begin{cases} a/2, & \text{if } a \equiv 0 \pmod{2}; \\ 3a+1, & \text{if } a \equiv 1 \pmod{2}, \end{cases} \qquad \forall a \in \mathbb{Z}.$$

For every $a \in \mathbb{Z}$, the infinite sequence $(C^n(a))_{n=0}^{\infty}$ obtained by iterating C is called a *Collatz sequence*. The Collatz conjecture asserts that every Collatz sequence starting with a positive integer contains 1. Note that once the term 1 appears, then the further terms are repetitions of the cycle $1 \to 4 \to 2 \to 1$. The Collatz conjecture is also known as the $3x + 1$ problem. The reader is referred to [1] and [2] for an annotated bibliography on this topic.

If the conjecture is false, then there must exist a Collatz sequence that has one of the following properties:

- It ends up with a cycle that does not contain 1;
- It has no bound.

In this paper, we give upper bounds for cycles in Collatz sequences consisting of positive integers in terms of the lengths of the cycles. As a corollary, we see that the number of such cycles of a given length is finite. Since an even number always becomes an odd number after a finitely many iteration by C, it suffices to consider the Collatz sequences starting with odd numbers. Accordingly, it is convenient to define a "shortcut" of the Collatz function which enables us to deal with only odd numbers. Put $\mathbb{Z}_{odd} := \mathbb{Z} \setminus 2\mathbb{Z}$, and define $f : \mathbb{Z}_{odd} \to \mathbb{Z}_{odd}$ by

$$f(a) := \frac{3a+1}{2^n}, \quad \forall a \in \mathbb{Z}_{odd}$$

where n is the multiplicity of the factor 2 in the number $3a + 1$. Then f is a function onto $\{a \in \mathbb{Z}_{odd} \mid a \equiv 1 \text{ or } 2 \pmod{3}\} = \{a \in \mathbb{Z} \mid a \equiv 1 \text{ or } 5 \pmod{6}\}$, but not one-to-one. Note that if a is a positive (respectively, negative), then so are all its descendants $f^i(a)$ $(i \in \mathbb{N})$. For a given $a \in \mathbb{Z}_{odd}$, if k is the minimum natural number with the property $f^k(a) = a$, then we call the sequence $(f^i(a))_{i=0}^{k-1}$ the *Collatz cycle starting with* a, and we call k its *length*. For instance, 1 is the length of the Collatz cycle starting with 1, which is the only known Collatz cycle with positive terms. We call a Collatz cycle is *positive* (respectively, *negative*) if the initial term is (hence, all terms are) positive (respectively, negative). The following is

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the theorem in this paper, which suggests that if a positive Collatz cycle of length k exists, then it is enough to operate the function f k-times to the natural numbers less than $k3^{k-1}$ to find out such a cycle.

Theorem 1. *If $a_{\min}$ is the minimum element of a positive Collatz cycle of length k (≥ 2), then $a_{\min} < k(3^k+1)/4 < k3^{k-1}$.*

Proof. Let $\Gamma := (a_1, \ldots, a_k)$ be the Collatz cycle of length k starting with $a_1 \in \mathbb{N}$. Without loss of generality, we may assume that $a_{\min} = a_k =: a$ is the minimum value in the cycle. $f^k(a)$ is explicitly written as

$$f^k(a) = \frac{3^k a + 3^{k-1} + 3^{k-2} \cdot 2^{n_1} + \cdots\cdots + 3 \cdot 2^{n_1+\cdots+n_{k-2}} + 2^{n_1+\cdots+n_{k-1}}}{2^{n_1+\cdots+n_k}} > \frac{3^k a}{2^{n_1+\cdots+n_k}}, \tag{1}$$

where n_i $(i \in \{1, \ldots, k\})$ is the multiplicity of the factor 2 in the number $3f^{i-1}(a)+1$. Note that the condition that $f^k(a) < f^{k-1}(a)$ (since $f^k(a) = a$ is the minimum) implies that $n_k \geq 2$. Setting $f^k(a) = a$ gives

$$a = \frac{\frac{3^{k-1}}{2^{n_1+\cdots+n_{k-1}}} + \frac{3^{k-2}}{2^{n_2+\cdots+n_{k-1}}} + \frac{3^{k-3}}{2^{n_3+\cdots+n_{k-1}}} + \cdots\cdots + \frac{3^2}{2^{n_{k-2}+n_{k-1}}} + \frac{3}{2^{n_{k-1}}} + 1}{2^{n_k}\left(1 - \frac{3^k}{2^{n_1+\cdots+n_k}}\right)}. \tag{2}$$

The restriction $f|_\Gamma$ of f to the cycle Γ is a one-to-one function onto itself, so one can consider its inverse $g := (f|_\Gamma)^{-1}$. Since $a_{i-1} = g(a_i) = (2^{n_{i-1}}a_i - 1)/3, \forall i \in \{2, \ldots, k\}$, it is easy to observe that for $i \in \{1, \ldots, k-1\}$,

$$\begin{aligned} a &< g^i(a) \\ &= \tfrac{2^{n_{k-i}+\cdots+n_{k-1}}a - 2^{n_{k-i}+\cdots+n_{k-2}} - 3\cdot 2^{n_{k-i}+\cdots+n_{k-3}} - \cdots\cdots - 3^{i-3}\cdot 2^{n_{k-i}+n_{k-i+1}} - 3^{i-2}\cdot 2^{n_{k-i}} - 3^{i-1}}{3^i} \\ &< \tfrac{2^{n_{k-i}+\cdots+n_{k-1}}}{3^i}a. \end{aligned}$$

Thus we have that

$$\frac{3^i}{2^{n_{k-i}+\cdots+n_{k-1}}} < 1, \quad \forall i \in \{1, \ldots, k-1\}. \tag{3}$$

Inequality (1) together with $f^k(a) = a$ gives that $3^k/2^{n_1+\cdots+n_k} < 1$,[1] which implies that

$$2^{n_1+\cdots+n_k} \geq 3^k + 1. \tag{4}$$

Applying Inequality (3) to the numerator and Inequality (4) together with the inequality $n_k \geq 2$ to the denominator of the right-hand side of Equation (2) yields the inequalities asserted. □

Corollary 2. *If $a_{\max}$ is the maximum element of a positive Collatz cycle of length $k \geq 1$, then*

$$a_{\max} < \frac{k}{6}\left(\frac{9}{2}\right)^k + \frac{2}{9}\left(\frac{7}{4}k - 1\right)\left(\frac{3}{2}\right)^k < \frac{4}{3}k\left(\frac{9}{2}\right)^{k-1}.$$

[1]The inequality $3^k/2^{n_1+\cdots+n_k} < 1$ can be also concluded from the fact that a must be positive using Equation (2).

Proof. The cycle (1) (the sequence consisting of only one term 1) is the only cycle of length 1 starting with a natural number, and in this case, the asserted inequalities hold. So we assume that $k \geq 2$. We use the same notation as one in the proof of Theorem 1. Let $i \in \{1, \dots, k-1\}$ be such that $a_{\max} = a_i = f^i(a)$. Note that the "possible" maximum value of $a_{\max}$ is obtained by setting $n_1 = \cdots = n_i = 1$. Thus

$$\begin{aligned} a_{\max} &\leq \tfrac{3^i a + 3^{i-1} + 3^{i-2}\cdot 2 + 3^{i-3}\cdot 2^2 \cdots\cdots + 3^2\cdot 2^{i-3} + 3\cdot 2^{i-2} + 2^{i-1}}{2^i} \\ &< \tfrac{3^i a + i3^{i-1}}{2^i} \leq \tfrac{3^{k-1}a + (k-1)3^{k-2}}{2^{k-1}}. \end{aligned}$$

Applying the first inequality of Theorem 1, we obtain the desired inequalities. $\square$

The following fact immediately follows from Corollary 2.

Corollary 3. *For each $k \in \mathbb{N}$, the number of the positive Collatz cycles of length k is finite.*

Remark 4. *In this paper, we gave bonds for positive Collatz cycles. Considering the Collatz cycle starting with a negative integer a is equivalent to considering the Collatz cycle starting with the positive integer $-a$ with the function f replaced by $\tilde{f}(a) := (3a-1)/2^n$, where n is the multiplicity of the factor 2 in the number $3a-1$. The difference between f and $\tilde{f}$ is just in the signs. However, this difference is crucial, and the techniques we used in this paper for positive Collatz cycles do not work for the negative ones, and hence we are unable to provide similar estimates. It is interesting to contrast with the fact that there are three known negative Collatz cycles, while there is only one known positive Collatz cycle (*[3]*).*

References

[1] J. C. Lagarias, *The $3x+1$ Problem: An Annotated Bibliography (1963–1999),* arXiv:math.NT/0309224
[2] J. C. Lagarias, *The $3x+1$ Problem: An Annotated Bibliography, II (2000–),* arXiv:math.NT/0608208
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